Chapter 4 Finite Elements

In the previous chapter an approach to solving differential equations using approximate integral methods was described. This scheme involved the selection of trial functions that must satisfy certain requirements for completeness. We will now demonstrate how the finite element method of specifying these trial functions leads to a flexible and computationally efficient numerical scheme.

4.1 The Finite Element Concept

In the finite element method the basis functions adopted are polynomials which are piecewise continuous over subdomains called finite elements. Nodes are located along the boundaries of each subdomain and each basis function is identified with a specific node. Depending on the forms of the bases, certain of their derivatives may also be continuous across element boundaries.

4.2 Linear Basis Functions

The basic features of the finite element concept can be easily visualized using the information illustrated in Fig. 4.1. Let the closed interval $x_1 < x < x_5$ represent the domain of interest and let this interval be subdivided by nodes into four subintervals or elements. The function $u(x)$ can be

represented approximately by the piecewise linear function $\hat{u}(x)$ which is defined by a series of the form

$$u \simeq \hat{u} = \sum_{j=1}^{N} u_j \phi_j(x) \tag{4.1}$$

where, for the linear scheme of Figs. 4.1 and 4.2 with five nodes, $N = 5$. The form of the functions $\phi_j(x)$, which are in fact the basis functions for this one-dimensional example, can be established by imposing appropriate conditions on $\hat{u}(x)$.

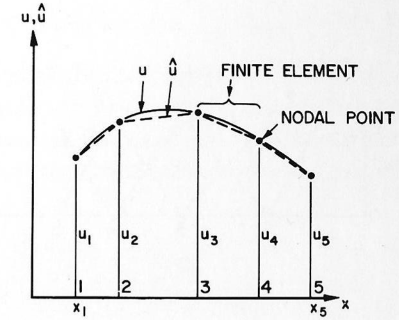

Fig. 4.1. One-dimensional finite element in global coordinates.

4.2.1 *Linear Basis Functions in Global Coordinates*

Let the interpolating polynomial $\hat{u}(x)$ be represented within each element by the general linear polynomial

$$\hat{u}(x) = a + bx \tag{4.2}$$

To solve for the two constants a and b, it is only necessary to require that $\hat{u}(x_i) = u_i$ and $\hat{u}(x_{i+1}) = u_{i+1}$, where x_i and x_{i+1} define the end points of the element. These constraints, in combination with (4.2), yield the matrix equation

$$\begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} = \begin{bmatrix} 1 & x_i \\ 1 & x_{i+1} \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix}$$

which can be solved for the coefficients

$$\begin{Bmatrix} a \\ b \end{Bmatrix} = \frac{1}{x_{i+1} - x_i} \begin{bmatrix} x_{i+1} & -x_i \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}$$

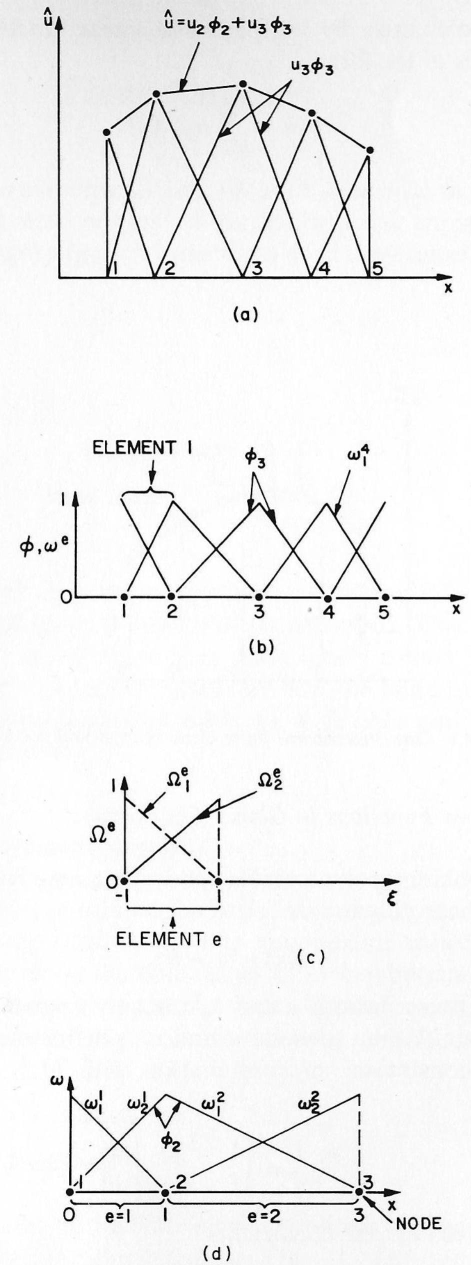

Fig. 4.2. The approximate function and corresponding coordinate functions.

The function $\hat{u}(x)$ may now be written for any element as

$$\hat{u} = \frac{x_{i+1}u_i - x_i u_{i+1}}{x_{i+1} - x_i} + \frac{u_{i+1} - u_i}{x_{i+1} - x_i} x$$

After rearrangement to obtain a form analogous to (4.1) this expression becomes

$$\hat{u} = u_i\left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right) + u_{i+1}\left(\frac{x - x_i}{x_{i+1} - x_i}\right) \tag{4.3}$$

Comparison of (4.3) and (4.1) demonstrates that the basis functions over the element are defined by

$$\phi_i = (x_{i+1} - x)/(x_{i+1} - x_i), \qquad \phi_{i+1} = (x - x_i)/(x_{i+1} - x_i)$$

In the formulation of these expressions for ϕ, only one element has been considered. Thus there are two basis functions that have nonzero components over each element of this example. The linear ϕ basis function is often called the chapeau function because it is hat-shaped when defined for a given node, i.e.,

$$\phi_i(x) = \begin{cases} (x - x_{i-1})/(x_i - x_{i-1}) & (x_{i-1} \le x \le x_i) \\ (x_{i+1} - x)/(x_{i+1} - x_i) & (x_i \le x \le x_{i+1}) \end{cases} \tag{4.4}$$

Note that basis function $\phi_i(x)$ has a value of unity at node i but goes to zero at all other nodes.

It should also be apparent at this point that the basis function ϕ_i has a component in each of two elements as illustrated in Fig. 4.2b. Basis functions, therefore, can be defined either by reference to the appropriate node or by reference to the element and the location within the element. For example, in element e, the two nonzero ϕ components may be denoted for the present case by

$$\phi_i(x_{i+1} \ge x \ge x_i) \equiv \omega_1^e(x) \equiv (x_{i+1} - x)/(x_{i+1} - x_i)$$
$$\phi_{i+1}(x_{i+1} \ge x \ge x_i) \equiv \omega_2^e(x) \equiv (x - x_i)/(x_{i+1} - x_i)$$

where the superscript e designates element number. Thus an equivalent statement of Eq. (4.1) is

$$\hat{u} = \sum_{e=1}^{E} \sum_{j=1}^{n} u_j^e \omega_j^e(x) \tag{4.5}$$

where E is the number of elements (four in the present example), and n the number of nonzero basis functions per element (two in the present example).

4.2.2 *Linear Basis Functions in Local Coordinates*

The basis functions are nearly always considered over one element at a time in practical application. Consequently, it is convenient to introduce a local coordinate system for each element to simplify integration. This is particularly true in the more complex element formulations where numerical integration is necessary. Linear basis functions defined in a local coordinate system are illustrated in Fig. 4.2c. Each element is defined such that ξ, the local coordinate, lies between -1 and 1. To distinguish basis functions defined in the local coordinate system from the basis functions in the global system, they are designated by Ω_i^e. Thus, for the present case, the expression for $\hat{u}$ over element e in local coordinates would be

$$\hat{u} = u_1^e \Omega_1^e + u_2^e \Omega_2^e \tag{4.6}$$

The appropriate expressions for the local basis functions are obtained as in the global coordinate system. A linear interpolation across the element is assumed, i.e.,

$$\hat{u}(\xi) = a + b\xi \tag{4.7}$$

and the constraints are imposed such that $\hat{u}(-1) = u_1^e$ and $\hat{u}(1) = u_2^e$. Solution of this equation for the constants a and b yields

$$\begin{Bmatrix} a \\ b \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$$

or

$$a = \tfrac{1}{2}(u_1^e + u_2^e), \qquad b = \tfrac{1}{2}(u_2^e - u_1^e)$$

Substitution of these expressions back into (4.7) and rearrangement yields

$$\hat{u}(\xi) = u_1^e[\tfrac{1}{2}(1 - \xi)] + u_2^e[\tfrac{1}{2}(1 + \xi)]$$

Comparison with Eq. (4.6) indicates that the linear basis functions in local coordinates are

$$\Omega_1^e = \tfrac{1}{2}(1 - \xi), \qquad \Omega_2^e = \tfrac{1}{2}(1 + \xi)$$

4.2.3 *Transformation Functions*

To integrate the basis functions in the local coordinate system, it is necessary to develop a means of transforming from global to local coordinates. The required transformation functions are obtained in a manner analogous to the development of the basis functions. In many cases, in fact, the basis and transformation functions are identical.

The development and application of coordinate functions will be presented through an example problem. Consider the one-dimensional finite element of Fig. 4.2d. In both the local and global coordinate systems, values of x and ξ will increase in direct proportion to the distance along the axes. Thus the transformation function will be of the form

$$x = a + b\xi \tag{4.8}$$

In addition, there must be a one-to-one correspondence between the node locations in x and in ξ. These constraints give rise to a set of equations analogous to those developed earlier for linear bases:

$$\begin{Bmatrix} x_1^e \\ x_2^e \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} a \\ b \end{Bmatrix} \tag{4.9}$$

Solution of (4.9) and substitution of the resulting values of a and b into (4.8) yields

$$x = \tfrac{1}{2}(x_2^e + x_1^e) + \tfrac{1}{2}(x_2^e - x_1^e)\xi$$

$\xi = \dfrac{x - \frac{1}{2}(x_2 + x_1)}{\frac{1}{2}(x_2 - x_1)}$

which can be rearranged to give

$\dfrac{d\xi}{dx} = \dfrac{2}{L}$ $\quad L = (x_2 - x_1)$

$$x = x_1^e[\tfrac{1}{2}(1 - \xi)] + x_2^e[\tfrac{1}{2}(1 + \xi)] \tag{4.10}$$

It is apparent from (4.10) that the functions required for transformation from the global to the local coordinate system are indeed the basis functions defined earlier. Moreover, Eq. (4.10) can be written over an element as

$$x = \sum_{j=1}^{2} x_j^e \Omega_j^e$$

While there is little, if any, conservation of effort in utilizing local coordinates in one dimension, a simple problem will be considered to demonstrate the general procedure involved in utilizing the transformation functions, and formulating the finite element solution procedure.

Example Consider the following equations and the finite element net of Fig. 4.2d:

$$Lu \equiv d^2u(x)/dx^2 = 3$$

with boundary conditions $u(0) = 1$ and $du(3)/dx = 0$.

Step 1 Formulate the approximate integral equations using the Galerkin method. From Chapter 3, we have

$$\langle L\hat{u} - f, \phi_i(x)\rangle = 0 \qquad (i = 1, 2, 3)$$

where, in this example, $f = 3$. Substitution of the governing differential equation yields

$$\langle (d^2\hat{u}/dx^2 - 3), \phi_i(x)\rangle = 0 \qquad (i = 1, 2, 3)$$

Application of integration by parts gives

$$\left\langle -\frac{d\hat{u}}{dx}, \frac{d\phi_i}{dx}\right\rangle - \langle 3, \phi_i\rangle + \frac{d\hat{u}}{dx}\phi_i\Big|_{x=0}^{x=3} = 0 \qquad (i = 1, 2, 3) \tag{4.11}$$

Step 2 Select basis functions for substitution into (4.11). Because the problem will be solved using linear bases, the appropriate expression for $\hat{u}$ in terms of these bases is

$$\hat{u}(x) = \sum_{j=1}^{3} u_j\phi_j(x) \tag{4.12}$$

Substitution of (4.12) into (4.11) yields

$$\sum_{j=1}^{3} u_j\left\langle \frac{d\phi_j}{dx}, \frac{d\phi_i}{dx}\right\rangle + \langle 3, \phi_i\rangle - \frac{d\hat{u}}{dx}\phi_i\Big|_{x=0}^{x=3} = 0 \qquad (i = 1, 2, 3)$$

Step 3 Formulate the system of algebraic equations. This is accomplished by writing in matrix form the equations generated successively by setting i to 1, 2, and 3:

$$\begin{bmatrix} \left\langle \frac{d\phi_1}{dx}, \frac{d\phi_1}{dx}\right\rangle & \left\langle \frac{d\phi_2}{dx}, \frac{d\phi_1}{dx}\right\rangle & 0 \\ \left\langle \frac{d\phi_1}{dx}, \frac{d\phi_2}{dx}\right\rangle & \left\langle \frac{d\phi_2}{dx}, \frac{d\phi_2}{dx}\right\rangle & \left\langle \frac{d\phi_3}{dx}, \frac{d\phi_2}{dx}\right\rangle \\ 0 & \left\langle \frac{d\phi_2}{dx}, \frac{d\phi_3}{dx}\right\rangle & \left\langle \frac{d\phi_3}{dx}, \frac{d\phi_3}{dx}\right\rangle \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} -\langle 3, \phi_1\rangle + \frac{d\hat{u}}{dx}\phi_1\Big|_{x=0}^{x=3} \\ -\langle 3, \phi_2\rangle + \frac{d\hat{u}}{dx}\phi_2\Big|_{x=0}^{x=3} \\ -\langle 3, \phi_3\rangle + \frac{d\hat{u}}{dx}\phi_3\Big|_{x=0}^{x=3} \end{Bmatrix} \tag{4.13}$$

Because $d\phi_1/dx$ and $d\phi_3/dx$ are never both nonzero in any part of the solution domain, zeros appear in the lower left and upper right corners of the coefficient matrix.

Step 4 Apply the boundary conditions. Because of the way the basis functions are defined, $u_1 = u\ (x = 0)$. Thus, to apply the condition that $u(0) = 1$, it is only necessary to delete the first row of Eq. (4.13) and replace it with the boundary condition. This is done by putting 1 in the first column, zeros in column two and three, and 1 in the top row of the right-hand side vector. Note that the term in the second row of the right-side vector $\phi_2\, d\hat{u}/dx|_{x=0}^{x=3}$ will also be zero because $\phi_2 = 0$ at $x = 0$ and $x = 3$. For this problem, the term in the third row of the right-side vector, $\phi_3\, d\hat{u}/dx|_{x=0}^{x=3}$ will also be zero because $\phi_3 = 0$ at $x = 0$ and, by the boundary condition, $d\hat{u}/dx = 0$ at $x = 3$. Thus Eq. (4.13) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ \left\langle \frac{d\phi_1}{dx}, \frac{d\phi_2}{dx}\right\rangle & \left\langle \frac{d\phi_2}{dx}, \frac{d\phi_2}{dx}\right\rangle & \left\langle \frac{d\phi_3}{dx}, \frac{d\phi_2}{dx}\right\rangle \\ 0 & \left\langle \frac{d\phi_2}{dx}, \frac{d\phi_3}{dx}\right\rangle & \left\langle \frac{d\phi_3}{dx}, \frac{d\phi_3}{dx}\right\rangle \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -\langle 3, \phi_2\rangle \\ -\langle 3, \phi_3\rangle \end{Bmatrix} \tag{4.14}$$

Note that although the coefficient matrix in (4.13) was symmetric, the coefficient matrix in (4.14) is nonsymmetric. Because great savings in computer storage can be achieved when the coefficient matrix is symmetric, it is advantageous to further rearrange (4.14) to a symmetric form. Because u_1 is known to be unity from the boundary condition, the coefficients for u_1 in the second and third rows can be moved to the right side of the matrix equation and replaced by a zero. Thus (4.14) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \left\langle \frac{d\phi_2}{dx}, \frac{d\phi_2}{dx}\right\rangle & \left\langle \frac{d\phi_3}{dx}, \frac{d\phi_2}{dx}\right\rangle \\ 0 & \left\langle \frac{d\phi_2}{dx}, \frac{d\phi_3}{dx}\right\rangle & \left\langle \frac{d\phi_3}{dx}, \frac{d\phi_3}{dx}\right\rangle \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -\langle 3, \phi_2\rangle - \left\langle \frac{d\phi_1}{dx}, \frac{d\phi_2}{dx}\right\rangle \\ -\langle 3, \phi_3\rangle \end{Bmatrix} \tag{4.15}$$

Because u_1 has now been completely uncoupled from the matrix problem, it is possible to condense the first row and first column out of the matrix and to solve the resulting 2×2 symmetric matrix problem enclosed in the dashed box.

Step 5 Transform the integrals to local coordinates and evaluate them element by element. Some of the terms in (4.15) will be nonzero in both elements and therefore must be integrated over both elements. Thus (4.15) becomes

$$\begin{bmatrix} \left\langle \frac{d\Omega_2^e}{dx}, \frac{d\Omega_2^e}{dx}\right\rangle^1 + \left\langle \frac{d\Omega_1^e}{dx}, \frac{d\Omega_1^e}{dx}\right\rangle^2 & \left\langle \frac{d\Omega_2^e}{dx}, \frac{d\Omega_1^e}{dx}\right\rangle^2 \\ \left\langle \frac{d\Omega_1^e}{dx}, \frac{d\Omega_2^e}{dx}\right\rangle^2 & \left\langle \frac{d\Omega_2^e}{dx}, \frac{d\Omega_2^e}{dx}\right\rangle^2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} -\langle 3, \Omega_2^e\rangle^1 - \langle 3, \Omega_1^e\rangle^2 - \left\langle \frac{d\Omega_1^e}{dx}, \frac{d\Omega_2^e}{dx}\right\rangle^1 \\ -\langle 3, \Omega_2^e\rangle^2 \end{Bmatrix} \tag{4.16}$$

where $\langle a, b\rangle^e$ denotes the integration of the product ab over element e. Now perform the integrations over element 1. The term $\langle d\Omega_2^e/dx, d\Omega_2^e/dx\rangle^1$ is easily evaluated after being rewritten in ξ coordinates:

$$\left\langle \frac{d\Omega_2^e}{dx}, \frac{d\Omega_2^e}{dx}\right\rangle^1 = \int_0^1 \frac{d\Omega_2^e}{dx}\frac{d\Omega_2^e}{dx}\, dx = \int_{-1}^1 \frac{d\Omega_2^e}{d\xi}\frac{d\xi}{dx}\frac{d\Omega_2^e}{d\xi}\frac{d\xi}{dx}\frac{dx}{d\xi}\, d\xi \tag{4.17}$$

Because $\Omega_2^e = \frac{1}{2}(1 + \xi)$, it is readily seen that $d\Omega_2^e/d\xi = \frac{1}{2}$. Derivatives of the form $dx/d\xi$ for element 1 can easily be obtained from Eq. (4.10):

$$x = x_1^1[\tfrac{1}{2}(1 - \xi)] + x_2^1[\tfrac{1}{2}(1 + \xi)]$$

Therefore

$$dx/d\xi = \tfrac{1}{2}(x_2^1 - x_1^1) = \tfrac{1}{2}(1 - 0) = \tfrac{1}{2}$$

Substitution back into (4.17) yields

dx/dξ

$$\langle d\Omega_2^e/dx, d\Omega_2^e/dx\rangle^1 = \int_{-1}^1 \tfrac{1}{2}(2)\tfrac{1}{2}(2)\tfrac{1}{2}\, d\xi = 1 \tag{4.18a}$$

The other required integrals over element 1 are also easily obtained:

$$\langle 3, \Omega_2^e\rangle^1 = \int_{-1}^1 3[\tfrac{1}{2}(1 + \xi)]\tfrac{1}{2}\, d\xi = \tfrac{3}{2} \tag{4.18b}$$

$$\langle d\Omega_1^e/dx, d\Omega_2^e/dx\rangle^1 = \int_{-1}^1 -\tfrac{1}{2}(2)\tfrac{1}{2}(2)\tfrac{1}{2}\, d\xi = -1 \tag{4.18c}$$

Substitution of (4.18) back into (4.16) yields

$$\begin{bmatrix} 1 + \left\langle \frac{d\Omega_1^e}{dx}, \frac{d\Omega_1^e}{dx}\right\rangle^2 & \left\langle \frac{d\Omega_2^e}{dx}, \frac{d\Omega_1^e}{dx}\right\rangle^2 \\ \left\langle \frac{d\Omega_1^e}{dx}, \frac{d\Omega_2^e}{dx}\right\rangle^2 & \left\langle \frac{d\Omega_2^e}{dx}, \frac{d\Omega_2^e}{dx}\right\rangle^2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{3}{2} - \langle 3, \Omega_1^e\rangle^2 + 1 \\ -\langle 3, \Omega_2^e\rangle^2 \end{Bmatrix} \tag{4.19}$$

The next step is to perform the integrations over element 2. For this element

$$x = x_1^2[\tfrac{1}{2}(1 - \xi)] + x_2^2[\tfrac{1}{2}(1 + \xi)]$$

and the transformation function is

$$dx/d\xi = \tfrac{1}{2}(x_2^2 - x_1^2) = \tfrac{1}{2}(3 - 1) = 1$$

Thus the integrals in (4.19) take the form

$$\left\langle \frac{d\Omega_1^e}{dx}, \frac{d\Omega_1^e}{dx}\right\rangle^2 = \int_{-1}^1 \frac{d\Omega_1^e}{d\xi}\frac{d\xi}{dx}\frac{d\Omega_1^e}{d\xi}\frac{d\xi}{dx}\frac{dx}{d\xi}\, d\xi$$

$$= \int_{-1}^1 \left(-\frac{1}{2}\right)(1)\left(-\frac{1}{2}\right)(1)(1)\, d\xi = \frac{1}{2} \tag{4.20a}$$

$$\left\langle \frac{d\Omega_1^e}{dx}, \frac{d\Omega_2^e}{dx}\right\rangle^2 = \int_{-1}^1 \frac{d\Omega_1^e}{d\xi}\frac{d\xi}{dx}\frac{d\Omega_2^e}{d\xi}\frac{d\xi}{dx}\frac{dx}{d\xi}\, d\xi$$

$$= \int_{-1}^1 \left(-\frac{1}{2}\right)(1)\left(\frac{1}{2}\right)(1)(1)\, d\xi = -\frac{1}{2} \tag{4.20b}$$

$$\left\langle \frac{d\Omega_2^e}{dx}, \frac{d\Omega_2^e}{dx}\right\rangle^2 = \int_{-1}^1 \frac{d\Omega_2^e}{d\xi}\frac{d\xi}{dx}\frac{d\Omega_2^e}{d\xi}\frac{d\xi}{dx}\frac{dx}{d\xi}\, d\xi$$

$$= \int_{-1}^1 \left(\frac{1}{2}\right)(1)\left(\frac{1}{2}\right)(1)(1)\, d\xi = \frac{1}{2} \tag{4.20c}$$

$$\langle 3, \Omega_1^e\rangle^2 = \int_{-1}^1 3\Omega_1^e\, dx/d\xi\, d\xi = \int_{-1}^1 3[\tfrac{1}{2}(1 - \xi)](1)\, d\xi = 3 \tag{4.20d}$$

$$\langle 3, \Omega_2^e\rangle^2 = \int_{-1}^1 3\Omega_2^e\, dx/d\xi\, d\xi = \int_{-1}^1 3[\tfrac{1}{2}(1 + \xi)](1)\, d\xi = 3 \tag{4.20e}$$

Substitution of the relations in (4.20) back into matrix equation (4.19) yields

$$\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{7}{2} \\ -3 \end{Bmatrix}$$

which has the solution $u_2 = -6.5$, $u_3 = -12.5$. It can be easily shown that the analytic solution to this problem is $u = \frac{3}{2}x^2 - 9x + 1$ and that the solutions obtained for u_2 and u_3 (as well as for u_1) are exact. The analytic solution is a quadratic function of x while in this example, the approximate solution was assumed to be piecewise linear. Therefore, if values of u are obtained by evaluating $\hat{u} = \sum_{j=1}^{3} u_j\phi_j$ at various locations on the x axis, there will be some error between the nodes.