Multiscale subgrid models of large eddy simulation for turbulent flows

Jianying Jiao and Ye Zhang
Department of Geology and Geophysics, University of Wyoming, Laramie, Wyoming, USA

Abstract

Purpose – The purpose of this paper is to propose three modified subgrid-scale (SGS) eddy-viscosity models to improve their original eddy-viscosity models (the Smagorinsky model (SM), the mixed-scale model (MSM), and the wall-adapted local eddy-viscosity model (WALE)) in the simulation of turbulent flows in near-wall region.

Design/methodology/approach – The eddy viscosity is related to the norm of strain rate tensor of the smallest resolved scales, instead of the norm of the resolved strain rate tensor of the large scales.

Findings – All the SGS viscosity of the modified eddy-viscosity models (small-large model, modified MSM, and modified WALE) is closer to \( y^{+3} \) behavior than those of the original eddy-viscosity models (SM, MSM, and WALE) near the wall.

Originality/value – The norm of strain rate tensor of the smallest scales used in eddy-viscosity models, instead of the norm of strain rate tensor, makes the eddy viscosity in near-wall region approach to zero in a physical sense.

Keywords Channel flow, Eddy viscosity, Square cylinder, Subgrid-scale model

Paper type Research paper

1. Introduction

With the development of computational fluid dynamics (CFD), large eddy simulation (LES) is commonly employed to investigate the instantaneous three-dimensional velocity structures of turbulent flows. Specially, many subgrid-scale (SGS) models of LES have been proposed, such as the classic Smagorinsky model (SM) (Smagorinsky, 1963), the mixed-scale model (MSM) (Sagaut, 1996), the wall-adapted local eddy-viscosity model (WALE) (Nicoud and Dukruc, 1999), the multiscale subgrid models (Hughes et al., 1998, 2000, 2001a, b), the dynamic eddy-viscosity models (Germano et al., 1991; Ghosal et al., 1995; Meneveau et al., 1996; Huang and Li, 2010; Lee et al., 2010; Singh and You, 2013), SGS models based on mixing length (Gu et al., 2011a, b; Zhao and You, 2012), scale similarity models (Liu et al., 1994; Davidson, 2004), vortex-based SGS models (White and Neph, 2008; Misra and Pullin, 1997; Chung and Matheou, 2014), and others. Most of the SGS models are variations of or improvements made upon the classic SM which tends to produce excessive dissipation in the near-wall region. In particular, SM gives non-zero eddy viscosity when fluid velocity gradient exists. However, turbulent fluctuations in the near-wall region are observed to be damped thus eddy viscosity there should be zero.

Among the SGS models cited above, the MSM (Sagaut, 1996) and the WALE (Nicoud and Dukruc, 1999) are of interest. Subgrid viscosity value of the MSM goes naturally to zero in the near-wall region, but MSM still predicts excessive dissipation in the same region. WALE produces a \( y^{+3} \) behavior in the near-wall region, but this model has deficiency in predicting the logarithmic region of channel flow (Nicoud and Dukruc, 1999). On the other hand, Hughes et al. (2001b) first introduced the variational multiscale model...
(VMS) within the framework of finite element method and subsequently proposed the VMS based on SM. The variational multiscale SGS model of (Hughes et al., 1998) is 
\[ \tau_{ij} = -2\nu_{sgs} \overline{S}_{ij} + \delta_{ij} \tau_{kk} / 3 \]
where the strain rate tensor of the smallest resolved scale \( \overline{S}_{ij} = (\partial (u_i - \overline{u}_i)/\partial x_j + \partial (u_j - \overline{u}_j)/\partial x_i) / 2 \) is obtained through test filtering. The variational multiscale SGS models have two possible variants, \( \nu_{sgs} = (C_s \Lambda)^2 \overline{|S|} \) or \( \nu_{sgs} = (C_s \Lambda)^2 S \), which are validated by Hughes et al. (1998, 2000, 2001a, b). This work develops improved MSM and WALE models based on the proposed theory of Hughes et al. (1998, 2000, 2001a, b).

Direct numerical simulations (DNS) of channel flows with different Reynolds numbers by Iwamoto et al. (2005) suggest that the near-wall turbulence is associated not only with the near-wall fine-scale structures, but also with the near-wall large scale structures. This means that subgrid viscosity should reflect the fine-scale fluctuations while the subgrid stress tensor in SM is associated with the large scale resolved strain rate tensor. Based on the idea of (Hughes et al., 1998, 2000, 2001a, b) and the results of Iwamoto et al. (2005), the norm of the large scale strain rate tensor of subgrid viscosity could be replaced by the norm of strain rate tensor of the smallest resolved scales. In the following, we substitute the norm of strain rate tensor of the smallest resolved scales into the above-mentioned eddy-viscosity models, such as MSM and WALE, and we evaluate the effect of this substitution through numerical simulations of (1) turbulent flows in a channel with a friction-velocity-based Reynolds number of 395 and 590, and (2) turbulent flows passing a square cylinder at a higher Reynolds number.

2. Eddy-viscosity models
The classical SGS model based on the eddy-viscosity assumption, or the SGS tensor, is as follows:

\[ \tau_{ij} = -2\nu_{sgs} \overline{S}_{ij} + \delta_{ij} \tau_{kk} / 3 \]  
(1)

where:

\[ \overline{S}_{ij} = (\partial \overline{u}_i / \partial x_j + \partial \overline{u}_j / \partial x_i) / 2 \]  
(2)

\( \overline{S}_{ij} \) and \( \overline{u}_i \) are the deformation tensor and velocity of the resolved field, respectively. In Smagorinsky's model, the eddy viscosity is assumed to be proportional to a subgrid characteristic length scale \( \Lambda \) and the norm of a local strain rate \( \overline{|S|} \). That is:

\[ \nu_{sgs} = (C_s \Lambda)^2 \overline{|S|} \]  
(3)

where \( C_s \) is Smagorinsky coefficient (take at a value of 0.1 (Smagorinsky, 1963)), \( \overline{\Lambda} \) is the grid filter scale, and \( \overline{|S|} = (2\overline{\overline{S}_{ij}} \overline{\overline{S}_{ij}})^{1/2} \).

In the MSM proposed by Sagaut (1996), the SGS viscosity is related not only by the norm of the strain rate tensor but also by the turbulence energy (the smallest resolved scales). The model then arises as a weighted geometric average of \( \overline{|S|} \) and SGS turbulence energy (\( q_{sgs} \)):

\[ \nu_{sgs} = C_{mf} \left( \overline{|S|} \right)^{\alpha} (q_{sgs})^{(1-\alpha)/2} \overline{\Lambda}^{1+\alpha} \]  
(4)

where \( q_{sgs} = 0.5 \overline{\overline{u}_i u_i} \) and \( \overline{u}_i = \overline{u}_i - \overline{\overline{u}_i} \) (\( \overline{u}_i \), \( \overline{u}_i \), and \( \overline{u}_i \) are the resolved smallest velocity, the filtered velocity, and the resolved velocity in the sense of the test filter, respectively).
\( \alpha \) is a weighting factor in the range of \((0, 1) \) and \( C_M \) is a constant. In this work, the values \( C_M = 0.02 \) and \( \alpha = 0.5 \) are used (Sagaut, 1996). As \( q_{\text{sgs}} \) is not immediately available, it is estimated by a filtering operation over the test filter \( \Lambda \) \((\Lambda = 2\Delta)\). Because of the manner in which \( q_{\text{sgs}} \) is determined, the model ensures a monotonic decay of the SGS viscosity as the wall is approached and vanishing eddy-viscosity level in non-turbulent conditions.

The WALE is proposed by Nicoud and Ducros (1999). In contrast to SM which relates the viscosity to the symmetric part of the velocity gradient tensor, \( \tilde{\bar{E}}_{ij} = \partial \bar{u}_i / \partial x_j \), the WALE model is based on the symmetric part of the square of this tensor, \( \bar{G}_{ij} = (\tilde{\bar{E}}_{ik} \tilde{\bar{E}}_{kj} + \tilde{\bar{E}}_{jk} \tilde{\bar{E}}_{ki})/2 \). Its traceless part, \( \bar{G}^a_{ij} = \bar{G}_{ij} - \delta_{ij} \bar{G}_{kk}/3 \), is used to determine the eddy viscosity using (Nicoud and Ducros, 1999):

\[
\nu_{\text{sgs}} = C_M \Lambda^{-2} \left( \sqrt{|G^a|} \right)^6 / \left( \frac{1}{3} + \left( \sqrt{|G^a|} \right)^5 \right)
\]

(5)

where \( C_M \) is a constant, which is assigned 0.1 in this work (Nicoud and Ducros, 1999).

An improvement of the SGS Smagorinsky model, the so-called small-large model (SLM), is proposed by Hughes et al. (2001b) based on the variational multiscale idea (Hughes et al., 1998, 2000, 2001a, b). The SLM is proposed as follow:

\[
\tau_{ij} = -2\nu_{\text{sgs}} \bar{S}_{ij} + \delta_{ij} \tau_{kk}/3
\]

(6)

\[
\nu_{\text{sgs}} = (C_s \Lambda)^2 \left| \bar{S}^a \right|
\]

(7)

\[
\left| \bar{S}^a \right| = \left( 2\bar{S}_{ij} S_{ij}^a \right)^{1/2}
\]

(8)

where the constant \( C_s \) is equal to 0.1 (Hughes et al., 2001b). The eddy viscosity of SLM still goes to a non-zero value in the vicinity of a wall, but the eddy-viscosity value of SLM is less than that of SM.

The subgrid viscosity of the MSM goes to zero in the near-wall region, because the small scale \( \tilde{u}_i \) goes to zero in the near-wall region. However, MSM still exhibits excessive dissipation in the near-wall region. In the following, eddy viscosity of a modified MSM (M-MSM) is proposed as follow:

\[
\nu_{\text{sgs}} = C_M \left( \left| \bar{S}^a \right| \right)^x \left( q_{\text{sgs}} \right)^{(1-x)/2-1} \Delta^{-1} + \alpha
\]

(9)

where \( \alpha \) is a weighting factor in the range of \((0, 1)\) and \( C_M \) is a constant. In this work, \( C_M \) of 0.02 and \( \alpha \) of 0.5, the same as those of MSM, are adopted for M-MSM. We will evaluate the effect of replacing the norm of the strain rate tensor \( S_{ij} \) in MSM with the norm of the strain rate tensor of the smallest resolved scales \( (S_{ij}^a) \).

Similarly, the norm of the strain rate tensor in the WALE is replaced by the norm of the strain rate tensor of the smallest resolved scales. Then, eddy viscosity of a modified
WALE (M-WALE) is proposed as follows:

$$\nu_{sgs} = C_m \Delta^2 \left( \sqrt{|\mathbf{G}^{xy}|} \right) / \left( |\mathbf{S}|^5 + \left( \sqrt{|\mathbf{G}^{yy}|} \right)^5 \right)$$ (10)

where:

$$\overline{G}^{xy}_{ij} = \overline{G}^{y}_{ij} \delta_{ij} \overline{G}^{x}_{kk} / 3$$ (11)

$$\overline{G}^{x}_{ij} = \left( \overline{G}^{x}_{ik} \overline{G}^{x}_{kj} + \overline{G}^{x}_{jk} \overline{G}^{x}_{ki} \right) / 2$$ (12)

$$\overline{g}^{s}_{ij} = \partial (\bar{u}_i - \bar{u}_j) / \partial x_j$$ (13)

where the constant $C_m = 0.1$ (Nicoud and Ducros, 1999).

To verify the validity of the modified SGS models, SM, MSM, WALE, SLM, M-MSM, and M-WALE are applied to simulating: channel flows with different friction-velocity-based Reynolds numbers; and turbulent flow past a square cylinder at a higher Reynolds number of 21,400.

OpenFOAM, a C++ code library of classes for developing CFD codes, is used in the present study. OpenFOAM discretizes the governing equations using a finite-volume approach (OpenFOAM, 2013), where the temporal term of the Navier-Stokes equation is discretized using a four-time-level scheme, which is 3rd order in time (Rossiello et al., 2007). Other terms of the Navier-Stokes equation are discretized using a second-order central difference. The governing equations are solved sequentially using the pressure-implicit-with-splitting-of-operator algorithm. Solution is performed implicitly using matrix inversion with the incomplete Cholesky conjugate gradient methods.

3. Channel flow

Using SM, MSM, WALE, SLM, M-MSM, and M-WALE, fully developed channel flows with a friction-velocity-based Reynolds number ($Re_\tau$) of 395 and 590 were simulated. Periodic boundary conditions are imposed in the stream-wise and span-wise directions. No-slip conditions are imposed at the solid walls. Parameters of the LESs for the channel flow are shown in Table 1. The reference data obtained with DNS by Moser et al. (1999) are used to verify those computed using the SGS models.

For the fully developed channel flow at $Re_\tau = 395$, mean velocities ($U^+$) and turbulence intensities computed by SM, MSM, WALE, SLM, M-MSM, and M-WALE are shown in Figure 1. Turbulence intensities are defined as root-mean-square (RMS) streamwise ($u_{rms}$), spanwise ($v_{rms}$), and normal ($w_{rms}$) velocity fluctuations. Mean velocities calculated by SLM, M-MSM, and M-WALE are closer to DNS's profile than those by SM, MSM, and WALE. $U^+$ computed by SM is underestimated, a result of excessive dissipation in the near-wall region. $U^+$ computed by MSM is more accurate.

<table>
<thead>
<tr>
<th>case</th>
<th>$Re_\tau$</th>
<th>$Re_\tau$</th>
<th>$L_x \times L_y \times L_z$</th>
<th>$N_x \times N_y \times N_z$</th>
<th>$\Delta x^+$</th>
<th>$\Delta y^+$</th>
<th>$\Delta z^+$</th>
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<tr>
<td>1</td>
<td>13,790</td>
<td>395</td>
<td>$\pi \delta \times 26 \times \pi \delta / 2$</td>
<td>$40 \times 50 \times 30$</td>
<td>31</td>
<td>0.7-53</td>
<td>21</td>
</tr>
<tr>
<td>2</td>
<td>22,000</td>
<td>590</td>
<td>$\pi \delta \times 26 \times \pi \delta / 2$</td>
<td>$50 \times 60 \times 40$</td>
<td>37</td>
<td>0.6-78</td>
<td>24</td>
</tr>
</tbody>
</table>

Table I. Problem setup of the large eddy simulations
Figure 1. The results of channel flow at Re_c = 395

Notes: (a) Mean velocities; (b) velocity fluctuations u_{rms}; (c) velocity fluctuations v_{rms}; (d) velocity fluctuations w_{rms}

than that of SM, because the turbulent energy (g_{sgs}) is more accurately represented in MSM, reducing excessive dissipation in the near-wall region. The mean velocity computed by WALE in the y^+ range of 20-100 is overestimated. The mean velocity computed by M-MALE is the most accurate among all the SGS models.

Figure 1(b) demonstrates that the peak values of the RMS stream-wise velocity fluctuations (u_{rms}) computed with all models are overestimated. The peak values computed by SLM, M-MSM, and M-WALE are a little larger than those of SM, MSM, and WALE, respectively. But, u_{rms} computed with SLM, M-MSM, and M-WALE are more accurate than those of SM, MSM, and WALE, respectively. All of the peak values of the span-wise velocity fluctuations (v_{rms}) are underestimated (Figure 1(c)). v_{rms} computed by SLM, M-MSM, and M-WALE are closer to the DNS data than those of MSM and WALE. However, v_{rms} by SM is in best agreement with the DNS data. Finally, the normal velocity fluctuations (w_{rms}) computed by SLM, M-MSM, and M-WALE are more accurate than those of MSM and WALE (Figure 1(d)). The peak w_{rms} values of SLM, M-MSM, and M-WALE are all underestimated but lie closer to that of DNS than those of SM, MSM, and WALE. The peak w_{rms} value of M-WALE is more accurate than those of SLM and M-MSM.

Figure 2 illustrates a set of results by SM, MSM, WALE, SLM, M-MSM, and M-WALE for channel flow at Re_c = 590, as compared with the DNS data. Figure 2(a)
illustrates that $U^+$ computed by SLM, M-MSM, and M-WALE are nearly identical and are more accurate than those of SM, MSM, and WALE. The velocity fluctuations $u_{rms}$ computed by all models at the $y^+$ range of 5-80 are overestimated (Figure 2(b)). The peak $u_{rms}$ values of M-MSM and M-WALE are closer to the DNS data than those of MSM and WALE, respectively. However, SM is the most accurate in capturing the peak $u_{rms}$. For all models, $v_{rms}$ and $w_{rms}$ are nearly identical to those computed earlier for channel flow at the $Re_x$ of 395 (Figure 2(c)-(d)).

Figure 3 shows dimensionless subgrid viscosities $(\nu_{s\infty}/\nu)$ of SM, MSM, WALE, M-MSM, SLM, and M-WALE for a channel flow at $Re_x = 395$. The dimensionless subgrid viscosities of SM, SLM, MSM, and M-MSM deviate from the theoretical wall-asymptotic behavior (i.e. $y^+$), while $(\nu_{s\infty}/\nu)$ of SLM and M-MSM are more accurate than those of SM and MSM in the near-wall region. The $(\nu_{s\infty}/\nu)$ of M-WALE is closer to the theoretical wall-asymptotic behavior ($y^+$) than that of WALE.

The above results indicate that: SLM, M-MSM, and M-WALE are more accurate than SM, MSM, and WALE in simulating wall-bounded turbulent flows; and, the subgrid viscosities of SLM, M-MSM, and M-WALE are closer to the theoretical wall-asymptotic behavior than those of SM, MSM, and WALE, respectively. This analysis suggests that the replacement of the norm of strain rate tensor by the strain
rate tensor of the smallest resolved scales, as implemented in SLM, M-MSM, and M-WALE, is an important improvement over SM, MSM, and WALE, respectively.

4. Flow past a square cylinder
In order to further validate the modified SGS models, turbulent flow past a square cylinder at a higher Reynolds number of 21,400 is simulated. Computational domain and coordinate definition for this problem are given in Figure 4. The Reynolds number involved in these new simulations is defined as \( \text{Re} = U_0 B / \nu \), where \( U_0 \) is the approaching free stream velocity and \( B \) is the side length of the three-dimensional square cylinder. As shown in this figure, the computational domain covers \( 21B \) in the stream-wise (\( x \)) direction (\( -6.5 < x/B < 14.5 \)), \( 9B \) in the lateral direction (\( -4.5 < y/B < 4.5 \)), and \( 6B \) in the vertical direction. At the inlet, a constant stream-wise velocity is specified, while all other velocities are set to zero. The outlet boundary condition is zero velocity gradient. The lateral boundary conditions are slipping. According to

![Figure 4. Geometrical model of flow past a square cylinder](image)
verification studies using wind tunnel experiments (Lyn et al., 1995), non-uniform grids of the computational domain must be used for mesh generation of the square cylinder. In this example, the domain is discretized using a $190 \times 80 \times 22$ grid in the streamwise, transversal, and vertical directions. The grid near the wall region is refined.

Time-averaged turbulent velocities and normal stresses past a square cylinder at a higher Reynolds number of 21,400 at different downstream locations ($x/B = 0.0, 1.0, 1.5, \text{ and } 5.0$) are presented in Figure 5. The time-averaged stream-wise velocities ($\langle U \rangle$) at these locations computed by SLM, M-MSM, and M-WALE are more accurate than those of SM, MSM, and WALE, respectively. ($U$) computed by SM is the least accurate. The time-averaged transverse velocities ($\langle V \rangle$) at $x/B = 1.0$ by SLM, M-MSM, and M-WALE are more accurate than those of SM, MSM, and WALE, respectively (Figure 5(b)). The time-averaged stream-wise normal stresses ($\langle u'^2 \rangle$) by SLM, M-MSM, and M-WALE are closer to the experiment data than those of SM, MSM, and WALE, respectively (Figure 5(c)). Finally, the time-averaged transverse normal stresses ($\langle v'^2 \rangle$) at $x/B = 1.0$ by SLM, M-MSM, and M-WALE are more accurate than those of SM, MSM, and WALE, but ($\langle v'^2 \rangle$) at $x/B = 1.5$ and 5.0 by SM, MSM, and WALE are more accurate than those of SLM, MSM, and M-WALE (Figure 5(d)).

Figure 6 illustrates that the time-averaged stream-wise velocities ($\langle U \rangle$) at centerline by SLM, M-MSM, and M-WALE are closer to the experiment data than

![Figure 5.](image-url)

**Notes:** (a) Time-averaged stream-wise velocity; (b) time-averaged transverse velocity; (c) time-averaged stream-wise normal stress; (d) time-averaged transverse normal stress
those of SM, MSM, and WALE, respectively. \( \langle U \rangle \) computed by M-WALE is more accurate than those of the other SGS models, while \( \langle U \rangle \) by SM is the least accurate.

5. Conclusions
A new set of modified SGS models, i.e., M-MSM and M-WALE, obtained by replacing the norm of strain rate tensor in the original eddy-viscosity models (MSM and WALE, respectively) with the norm of strain rate tensor of the smallest scales, is proposed. By simulating channel flows at a \( \text{Re}_\tau \) of 395 and 590 and comparing the results against DNS, the new models are more accurate than the original eddy-viscosity models. Specifically, SLM, M-MSM, and M-WALE are more accurate than SM, MSM, and WALE, respectively. In the near-wall region, SGS viscosities computed by SLM, M-MSM, and M-WALE are closer to the expected \( y^+3 \) behavior than the original models. Moreover, results simulating turbulent flow past a square cylinder at a higher Reynolds number of 21,400 suggest that the new models are again more accurate, as they can more reasonably capture subgrid viscosity in the near-wall region. This suggests that the norm of the strain rate tensor of the smallest scales used in the new models makes eddy viscosity in the near-wall region approach zero in a physical sense.

References


**Corresponding author**
Jianying Jiao can be contacted at: jiaojianying@gmail.com

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