Problem 2, Mon. Sept. 11 2017 Due Fri. Sept. 22 Name:_________________________

Exponential Decay

Where did we get the expression $[^{235}U] = [^{235}U]_0 e^{-\lambda t}$ to describe the radioactive decay of ^{235}U over time? ([^{235}U] means "the concentration of ^{235}U", [^{235}U](t) means "the concentration of ^{235}U as a function of time", $[^{235}U]_0$ means "the concentration of ^{235}U at time = zero – i.e., at the beginning", λ is a rate constant, and t is time. Of course, this equation applies not just to the decay of ^{235}U, but to what we call the "first-order decay" of many things, including radioactive isotopes generally.

The equation $[^{235}U] = [^{235}U]_0 e^{-\lambda t}$ follows from an extremely simple rate law:

$$\frac{d[^{235}U]}{dt} = -\lambda[^{235}U] \qquad 1$$

First, because [^{235}U] is *decreasing* because of decay, we know that this rate is a negative number. Second, the equation just means that the rate of change of [^{235}U] with time is directly proportional to the concentration of ^{235}U ([^{235}U]). Any given atom of ^{235}U has the same chance of decay as any other atom in a given time period. The more atoms you have, the more atoms decay in a given time period. That's all we're saying here. Equation 1 is a very simple differential equation, which can be solved by algebraic rearrangement:

$$\frac{d[^{235}U]}{[^{235}U]} = -\lambda dt \qquad 2$$

Now, this isn't a math class, so I'm not going to tell you why this is true, but $\int dN/N = \ln N + c$ (where c is a constant of integration). So, we can perform an indefinite integral on both sides to obtain:

$$\ln[^{235}U] = -\lambda t + C \qquad 3$$

where C is another constant of integration (we'll get a constant of integration on both sides, but we can consolidate the two into a single constant on whatever side we choose). We have to define what we call "boundary conditions" to figure out what C is. A useful boundary condition arises by asking "how much ^{235}U is there at the beginning, when t = 0?" When t = 0, C = $[^{235}U]_0$. In other words, at t = 0 we have as much ^{235}U as we started with (duh). So, we can write

$$\ln[^{235}U] = -\lambda t + \ln[^{235}U]_0 \qquad 4$$

because the identity of C doesn't change just because *t* does not equal zero – it's a constant, after all. We rearrange to obtain:

$$\ln[^{235}U] - \ln[^{235}U]_0 = -\lambda t \qquad 5$$

This is the same as:

$$\ln\left(\frac{[^{235}U]}{[^{235}U]_0}\right) = -\lambda t \qquad 6$$

Now, remembering that $e^{\ln(x)} = x$, we have $\frac{[^{235}U]}{[^{235}U]_0} = e^{-\lambda t}$

$$[^{235}U] = [^{235}U]_0 e^{-\lambda t} \qquad 7$$

Putting this solution into words, this simply says that the concentration of ^{235}U decreases with time from $[^{235}U]_0$ (the amount we started with) to zero (eventually) in an exponential fashion. The rate is controlled by the rate constant λ, which you encounter in lab this week. Remember that although we derived this with ^{235}U, you can substitute ^{238}U in there because it follows the same mathematical rules.

Problem: Using the result of the above derivation (equation 7), and the information in the table below, **calculate the half-life of ^{235}U and ^{238}U from the decay constant (λ is the decay constant)**. I've already given you the half-life below – I'm just asking you to show me how you calculate the number in the half-life column from the numbers in the decay constant column:

Isotope	Abundance	half-life (y)	decay constant (y^{-1})
^{238}U	99.2743%	4.468×10^9	1.55125×10^{-10}
^{235}U	0.7200	0.7038×10^9	9.8485×10^{-10}

Problem 2: When one half-life of ^{235}U has passed, what percentage of the original ^{238}U (NOT ^{235}U) in the rock remains?

Problem 3: If the Earth is 4.6 billion years old, is the ratio of parent ^{238}U to daughter product (^{238}U decays through a reaction chain to a stable daughter product, ^{206}Pb) slightly greater than or slightly less than 1?